Instabilities

and the Connection to Pattern Formation

I. General Aspects

II. A Criterion for the Formation of Complex Systems

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Preface to General Aspects

Within this paper I try to show the similarities that exist aswell in the *Theory of Hydrodynamic Instabilites* and in the *Theory of Complex Systems*. Both are trying to find out under which circumstances the system starts to behave in complex way. They use different set of vocabulary but use similar mathematical tools.

Instabilites are very important effects that give rise to a number of phenomena which are essentially connected to life itself. The weather or the magnetosphere of the earth are only possible because instabilities exists.

My overall aim is to find a general description to such complex systems as life, which is the most complex one among all complex systems we know. Still there exists no satisfying definition of life from the physical point of view. By investigating complex systems I think one is on the right way to understand *what is life*.

I want to thank Helfried Biernat for supporting and motivating me and giving me the possibility to prepare this talk for his lecture.

Florian Wodlei Klagenfurt, am 24.1.2009

Preface to A Criterion for the Formation of Complex Systems

After I gave my first talk about the *General Aspects* between (magneto)hydrodynamic instabilities and dissipative systems I wanted to find a criterion when such complex systems occur. With the help of the book by Nicolis and Prigogine *Selforganization in Nonequilibrium Systems* [2] I could imagine how such a criterion should look like. My progress in understanding this criterion is the topic of this paper. For a better understanding I also suggest to read my paper about *Entropy and Pattern Formation in Complex Systems*

Again thanks to the inestimable support by Helfried Biernat, who gave me the possibility to prepare this talk for his lecture.

Florian Wodlei Graz, am 18.6.2009

1

General Aspects

1.1. Selected Examples of Instabilities

I want to give here some examples of instabilities which are in contrast to the Rayleigh-Taylor or the Kelvin-Helmholtz instabilities normally mentioned within the context of *Complex Systems*. That are the Rayleigh-Bénard instability and a more complicated system the Belousov-Zhabotinsky reaction.

1.1.1. The Rayleigh-Bérnard Instability

The Rayleigh-Bénard instability is a thermal hydrodynamic instability that forms in horizontal layers of fluid heated from below (see figure 1). If the temperature difference ΔT between the heated bottom and the upper layer exceeds a critical value ΔT_c , the layer in between exhibits a bulk motion, which is called *thermal convection*.

The first sophisticated experiment was made by Bénard in 1900, though the phenomenon of thermal convection itself had been recognized earlier by Count Rumford in 1797 and James Thomson in 1882.

Rayleigh showed that what decides the stability is the numerical value of the nondimesional parameter, which is called the *Rayleigh number*

$$R = \frac{g\alpha\Delta T d^3}{\kappa\nu} \tag{1.1}$$

Where g denotes the acceleration due to gravity, ΔT the temperature difference d the depth of the layer and α, κ and ν are the coefficient of volume expansion, thermometric conductivity and



(a) The RB-instability in a layer (b) magnified part of the RB-instability of the left picture. Note the hexagonal of aluminium shape of the convection cells

Figure 1.1: The Rayleigh-Bénard instability

kinematic viscosity, respectively. Rayleigh showed that for a critical $R = R_c$ macroscopic patterns appear.

1.1.2. The Belousov Zhabotinsky Reaction

I would add the Belousov Zhabotinsky reaction to the type of *(chemical) reaction instabilities.* The Belousov Zhabotinsky reaction was first discovered by Belousov a russian scientist in 1950 but the importancy of this reaction was not recognized by the scientific community. When Zhabotinsky started the investigation of the reaction around 1960 the work was taken more serious also outside the soviet union.

Ilya Prigogine got his nobel prize for his work on the topic of complex systems where he also investigated the Belousov Zhabotinsky reaction.

The Belousov Zhabotinsky reaction is a liquid chemical reaction with almost only anorganic components. All of them not very exotic like sulfuric acid, sodium bromate, sodium bromide and as organic compound, malonic acid and ferroin. If put together in a arbitrary way nothing will happen, the solution remains homogeneous. There exists a critical start concentration set



(a) The BZ Reaction realized on a plate (b) magnfied part of the BZ reaction from left picture. Note the spirallike behaviour

Figure 1.2: The Belousov Zhabotinsky reaction

$$\{c_i^o\} = \{c_{i,crit}^o\}$$
(1.2)

with c_i^o the start concentrations of the components, for which the solution starts to form macroscopic patterns (like those shown in figure 2).

1.2. What is a Pattern?

In the everyday life it seams not to be difficult to identify a pattern, like the pattern of the clouds in the sky or the pattern of the coffee ground. A physical definition of it on the other hand is more difficult to find.

If an object shows a pattern that means that there exists a *symmetry*. Symmetries are very important in theoretical physics since to every symmetry there exists a conservation law (Emmy Noether [1], [2]).

But what exactly is a symmetry? The symmetry is a phenomenon under which an object remains unchanged if a transformation is applied. In figure 3 I give some examples of objects which show symmetries.

Now that we know what is a pattern we can make the connection to complex systems or in other words (hydrodynamic) instabilities. If we look at figure 1 and 2 we see pattern. In figure 1



Figure 1.3: Examples of symmetries. (a) a hexagonal symmetry as example for translational symmetry. (b) a distorted hexagonal symmetry as another example for translational symmetry. (c) cylindrical symmetrie with discrete radial symmetry. (d) cylindrical symmetry with geometric radial symmetry. (e) The Sierpinski triangle as an example of scale or fractal symmetry. (f) another example of scale or fractal symmetry

the thermal convection cells show hexagonal symmetry. In figure 2 the wave fronts show cylindrial symmetry.

1.3. The Connection between Instability and Pattern Formation

1.3.1. The Rayleigh-Bénard Instability

For simplicity we will discuss here a simple Rayleigh-Bénard setup where we have a experimental setup like shown in figure 4. Here we have two plates which are cylindric to the y-axis and between the plates we have fluid. Where at the start the plates have the same temperature $(T_1 = T_2)$ and for that reason the fluid is in a homogeneous state. The homogenity of the system extends to all its properties, and in particular to its temperature. Such a state is usually called an *equilibrium state* of the system.

If we would now apply a *local* temperature rise on one of the plates the system would not feel effected by this so called *perturbation* and would return to the equilibrium state. By heating the fluid layer from below we give rise to a temperature gradient of ΔT . This state where the temperature gradient is not very big is called *thermal conduction* and the properties along this gradient differs practically in linear fashion. As long as this external constraints exists, the state is not in equilibrium but stable.

If we remove the system farther and farther from equilibrium by increasing ΔT , suddenly at a critical value ΔT_c the system becomes *unstable*¹. If now a perturbation² is applied the system is not coming back to equilibrium but enters a new state which is usually called a *far from equilibrium state*. In this state a number of new phenomena occurs. The fluid begins to perform a bulk movement which is far from random, in our terms we would call it a *pattern*. The fluid is now structured in a series of small convection cells (see figure 4) known as *Bénard cells*.

1.3.2. The Belousov Zhabotinsky Reaction

The Belousov Zhabotinsky reaction is a hydrodynamical system in which chemical reactions occur.

In contrast to pure hydrodynamical systems, where the forces only come from the external forces the Belousov Zhabotinsky reaction shows effects due to the fact that chemical reactions occur. A chemical reaction is usually described by a reaction equation like

$$A + B \rightleftharpoons C + D \tag{1.3}$$

 $^{^1}$ This unstable behaviour at the critical value ΔT_c is responsible that this phenomenon has the name *instability*

²This perturbations are usually the internal fluctuations generated by the system itself



Figure 1.4: Formation of convection cells known as Bénard cells

Where the left arrow describes the backreaction. For the Belousov Zahbotinsky Reaction we have a complicated set of reaction equations. The Belousov Zahbotinsky reaction can be seen as a three step cycle. The first step is given by

$$BrO_3^- + 5Br^- + 6H^+ \to 3Br_2 + 3H_2O$$
 (1.4)

The intermediate $HBrO_2$ begins reaction with bromate as the step one slows done. This is the beginning of the second step which is given by

$$BrO_{3}^{-} + HBrO_{2} + 2Ce^{3+} + 3H^{+} \rightarrow 2HBrO_{2} + 2Ce^{4+} + H_{2}O$$
 (1.5)

The intermediate radical molecule BrO_2 is rapidly oxidizing the catalyst (Ce^{3+}) and turning into a $HBrO_2$. Thus leading to an increase in $HBrO_2$. This allows autocatalysis. As step one and step two removes bromide ion and oxidize the catalyst, the last step of the cycle has to come to the start by producing bromide ion and bringing the catalyst in its reduce form. This can be decribed by

$$2Ce^{4+} + BrCH(CO_2H)_2 + CH_2(CO_2H)_2 \rightarrow f Br^- + 2Ce^{3+} + other \ products$$
 (1.6)

where f is an unknown stoichiometric coefficient. For the spatial pattern formation like shown in figure 5 we also need to add a diffusion term to the corresponding equations.

If we have this cycle the complete system is at that point already in a nonequilibrium state. How can one describes the step from equilibrium to nonequilibrium for this reaction? In the Bénard problem we were able to move the system to a nonequillibrium state by applying a small temperature gradient - the equivalence here would be to bring the system to a state where it can start this cyclic behavior. Can we expect from such a system a behaviour like in the Benard problem? Here the criticality arises from the start concentrations of the constituents. Because of the criticality of the concentrations, the system shows after a few seconds complex behaviour in forming cylindrical waves (see figure 5).



Figure 1.5: Formation of cyclic and spirallike structures in the Belousov Zahbotinsky reaction

1.3.3. The hydrodynamic Instabilites

From the viewpoint of complex systems the hydrodynamic instabilites are also complex systems. Here we make the connection to the above described systems.

1.3.3.0.1. The Rayleigh-Taylor Instability is an interface instability which occurs between two fluids when the lighter fluid is pushing the heavier fluid. The effect is that so called "fingers" form (see figure 6a/b). One can find that if

$$\rho_2 > \rho_1 \tag{1.7}$$

The system remains in a nonequillibrium state and if

$$\rho_2 < \rho_1 \tag{1.8}$$

the system is unstable for all disturbance with wavenumbers smaller then

$$k_{crit.} = \sqrt{\frac{(\rho_2 - \rho_1)g}{\gamma}}$$
(1.9)

1.3.3.0.2. The Kelvin Helmholtz Instability is also an interface instability which occurs between two fluids when one fluid is flowing over another. The effect is the evolution of spirals on the interface (see figure 6c/d).

Also here one can find a criterion under which circumstances the system is unstable. For all (horizontal) disturbances with wavenumber bigger then

$$k_{crit.} = \frac{g}{U^2} \frac{\rho_1^2 - \rho_2^2}{\rho_1 \rho_2}$$
(1.10)

where g denotes the acceleration due to gravity, U is the velocity in x-direxction and ρ_i being the densities of the two layers, the system is unstable.



(b) A simulation of the Rayleigh-Taylor instability where as a result of the induced velocity gradient also Kelvin-Helmholtz instabilities occur (details see [5])

Figure 1.6: The Rayleigh-Taylor and the Kelvin-Helmholtz instability

1.4. Multistability and dissipative Structures

Towards a general Formulation of Complex Systems

Like Prigogine suggests the domain of the *dissipative structures*, the stable complex systems, is the nonlinear branch of nonequilibrium thermodynamics. Only far from equilibrium such structures are able to occur.

Thermodynamisc should be classified like in table 1.

Thermostatics	linear Thermodynamics	nonlinear Thermodynamics						
i.e. Thermodynamics	i.e. nonequillibrium Thermodynamics	i.e. far from equilibrium Thermodynamic						
equilibrium systems	diffusion, heat conduction	dissipative structures						
quasistatic processes		hydrodynamic instabilites complex systems						

Table 1.1: classification of "thermodynamics"

1.4.1. Multistability

What is usually studied in the theory of hydrodynamic Instabilities is the reaction of system to a perturbation and moreover the conditions when a system switches from a nonequilibrium state to an far from nonequilibrium state where patterns occur.

In the vocabulary of complex systems this state would be called a *dissipative system*. The term *instability* then is nomore fitting to this phenomena because the phenomena seams to be stable. To make it more general one should talk in terms of *Multistability*.

This would include the transition from a (meta)stable nonequilibrium state to a stable far from equilibrium state - a dissipative structure. The term *instability* is then ment in the sense that the (meta)stable state becomes instable and turns to another stable state, the far from equilibrium state.

1.4.2. Why dissipative structures can only occur far from equilibrium? The Minimum Entropy Production Theorem

In the linear thermodynamical regime the minimum entropy production theorem drives the system back to a steady state (i.e in the Bénard problem the system would if exposed to an perturbation return to the thermal conduction state) so that its impossible to form dissipative structures.

1.4. MULTISTABILITY AND DISSIPATIVE STRUCTURES

The entropy production \dot{S} plays the role of the thermodynamic potentials (i.e. the Gibbs energy G, the enthalpy H,...) in the equilibrium thermodynamic.



Figure 1.7: The stability of nonequibrium states (where the abscissa is a variable describing the system).

2

A Criterion for the Formation of Complex Systems

2.1. Introduction

Compared to the last chapter this chapter is more general and the theory developped here applies to all the systems discussed in last chapter.

Here I am going to derive a general criterion for the formation of complex systems which forms beyond the linear range, i.e. beyond the *Theorem of Minimum Entropy Production*. Therefore we need the branch of Nonequilibrium Thermodynamics. In the following sections I am going to introduce the needed terms and definitions.

2.2. Irreversible Processes in Thermodynamics

From the second theorem of Carnot it follows that for an isolated system (i.e. $\delta Q = 0$) in which irreversible processes occur the following equation is true

$$S_f - S_i \ge 0 \tag{2.1}$$

This is sometimes considered as an alterntive formulation of the *second principle of thermodynamics.* It says that in an isolated system the entropy does not change during a reversible process. On the other hand it says that in an isolated system the entropy increases during an irreversible process.

2.3. Nonequilibrium Thermodynamics

2.3.1. Local Formulation in Nonequilibrium Thermodynamics

The system of our interest are neither in equilibrium nor isolated. The systems which we adress are biological systems. From the thermodynamical standpoint those systems are *open systems*, which constantly exchange energy and matter with the environement.

In such a situation it is convenient to work with a local macroscopic description of matter. We define the entropy density s_v as

$$S = \int s_v dV \tag{2.2}$$

the entropy is then a local function of the state variables ρ_i , i.e. the concentrations of the chemical components of the system. This concentrations vary in space and time.

$$s_v = s_v[\rho_1(\boldsymbol{r}, t), \rho_2(\boldsymbol{r}, t), \dots, \rho_n(\boldsymbol{r}, t)]$$
(2.3)

2.3.2. Entropy Production in Nonequilibrium Systems

From the standpoint of entropy an open system is producing and exchanging entropy with the environement during the time (see figure 1). That means after a time period the total entropy





changes like

$$dS = d_e S + d_i S \tag{2.4}$$

To calculate the entropy production we first calculate the time derivative of the entropy density

$$\frac{\partial s_v}{\partial t} = \sum_i \frac{\partial s_v}{\partial \rho_i} \frac{\partial \rho_i}{\partial t}$$
(2.5)

with a local formulation of Gibbs fundamental equation we get

$$\frac{\partial s_v}{\partial \rho_i} = -\frac{\mu_i}{T} \tag{2.6}$$

Then we have for the time derivative of the entropy density

$$\frac{\partial s_v}{\partial t} = -\sum_i \frac{\mu_i}{T} \frac{\partial \rho_i}{\partial t}$$
(2.7)

To get an expression for $rac{\partial
ho_i}{\partial t}$ we investigate the mass-balance equation.

2.3.2.1. Mass-Balance Equation

The total mass change in an open system is given by the internal reactions and the exchange with the environement

$$\frac{dm_{X_j}}{dt} = \frac{d_i m_{X_j}}{dt} + \frac{d_e m_{X_j}}{dt}$$
(2.8)

The term $\frac{d_e m_{X_j}}{dt}$ is given by the penetration of mass through the surface boundary Σ and yields

$$rac{d_e m_{X_j}}{dt} = -\int_{\Sigma} \boldsymbol{j}_{X_j}^{\Sigma} \cdot \boldsymbol{n} \ d\Sigma$$
 (2.9)

The term $rac{d_e m_{X_j}}{dt}$ is given by the internal chemical reaction 1 of the open system.



Figure 2.2: open system consisting of the chemical constituents X_i exchanging the fluxes $j_{X_n}^{\Sigma}$ with the environement

¹the mass which is created in the system can also be created by other processes not explicit by chemical reactions. For example it could be created by radioactive transformations or elementary particle processes - still the description is analogous

Suppose we have a set of chemical reactions in a small volume ΔV of the system

$$A + X_j \xrightarrow{k_1} A + B, \qquad A + 2X_j \xrightarrow{k_2} A + B + C$$
 (2.10)

where k_1 is the rate constant of the reaction. Then the reaction rate (change of a the mass of a constituent per time) for the constituent X_j is given by

$$W_1 = -k_1 \rho_A \ \rho_{X_j} \Delta V, \qquad W_2 = -k_2 \rho_A \ \rho_{X_j}^2 \Delta V$$
 (2.11)

The stoichiometric coefficients of the constituent in this reactions are 2 and 1. Therefore the mass change in the volume ΔV of the constituent X_i is given by

$$\frac{d_e m_{X_j}}{dt} = -2k_1 \rho_A \ \rho_{X_j} \Delta V - 2k_2 \rho_A \ \rho_{X_j}^2 \Delta V \tag{2.12}$$

General for a arbritrary reaction we would have for the total mass change in the volume ΔV of the constituent X_j is

$$\frac{d_e m_{X_j}}{dt} = \sum_r \nu_{jr} W_r \tag{2.13}$$

where ν_{jr} is the stoichiometric constant of the constituent X_j involved in the reaction r and summed over all reaction. If we introduce the *reaction rate per unit volume* w_r we have

$$\frac{d_e m_{X_j}}{dt} = \sum_r \nu_{jr} \int w_r \ dV \tag{2.14}$$

Summing up and inserting the identity $m_{X_j} = \int \rho_j \ dV$ we get the total mass change of the constituent X_j

$$\frac{d}{dt} \int \rho_{X_j} \, dV = \sum_r \nu_{jr} \int w_r \, dV - \int_{\Sigma} \boldsymbol{j}_{X_j}^{\Sigma} \cdot \boldsymbol{n} \, d\Sigma$$
(2.15)

With the Gaussian divergence theorem we get

$$\frac{d}{dt} \int \rho_{X_j} \, dV = \sum_r \nu_{jr} \int w_r \, dV - \int \boldsymbol{\nabla} \cdot \boldsymbol{j}_{X_j} \, dV \tag{2.16}$$

a local equation for ho_{X_j} is obtained by

$$\frac{\partial \rho_{X_j}}{\partial t} = \sum_r \nu_{jr} w_r - \boldsymbol{\nabla} \cdot \boldsymbol{j}_{X_j}$$
(2.17)

This is the required equation. Now we can come back to the equation (2.7). Inserting (2.17) in (2.7) we get

$$\frac{\partial s_v}{\partial t} = -\sum_j \frac{\mu_j}{T} \left(\sum_r \nu_{jr} w_r - \boldsymbol{\nabla} \cdot \boldsymbol{j}_{X_j} \right)$$
(2.18)

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2.3. NONEQUILIBRIUM THERMODYNAMICS

$$\frac{\partial s_v}{\partial t} = -\sum_j \sum_r \frac{\mu_j}{T} \nu_{jr} w_r + \sum_j \frac{\mu_j}{T} \boldsymbol{\nabla} \cdot \boldsymbol{j}_{X_j}$$
(2.19)

We now introduce the chemical affinity \mathscr{A}_r of the reaction r as

$$\mathscr{A}_r = -\sum_j \mu_j \nu_{jr} \tag{2.20}$$

and we get

$$\frac{\partial s_v}{\partial t} = \sum_r \frac{\mathscr{A}_r}{T} w_r + \sum_j \frac{\mu_j}{T} \boldsymbol{\nabla} \cdot \boldsymbol{j}_{X_j}$$
(2.21)

With the following identity

$$\boldsymbol{\nabla}\left(\frac{\mu_j}{T}\boldsymbol{j}_{X_j}\right) = \boldsymbol{j}_{X_j} \cdot \boldsymbol{\nabla}\frac{\mu_j}{T} + \frac{\mu_j}{T}\boldsymbol{\nabla} \cdot \boldsymbol{j}_{X_j}$$
(2.22)

we can further rewrite the last equation to

$$\frac{\partial s_v}{\partial t} = \sum_r \frac{\mathscr{A}_r}{T} w_r + \sum_j \nabla \left(\frac{\mu_j}{T} \boldsymbol{j}_{X_j}\right) - \sum_j \boldsymbol{j}_{X_j} \cdot \nabla \frac{\mu_j}{T}$$
(2.23)

If we define the entropy flux density as $oldsymbol{J}_s = -\sum_j rac{\mu_j}{T} oldsymbol{j}_{X_j}$ we get

$$\frac{\partial s_v}{\partial t} = \sum_r \frac{\mathscr{A}_r}{T} w_r - \sum_j \boldsymbol{j}_{X_j} \cdot \boldsymbol{\nabla} \frac{\mu_j}{T} - \boldsymbol{\nabla} \boldsymbol{J}_s$$
(2.24)

Bringing the last term to the left hand side we get

$$\frac{\partial s_v}{\partial t} + \nabla \boldsymbol{J}_s = \sum_r \frac{\mathscr{A}_r}{T} w_r - \sum_j \boldsymbol{j}_{X_j} \cdot \nabla \frac{\mu_j}{T}$$
(2.25)

We can identify the left hand side as the total derivative of the entropy density with respect to time, and that is the reason why the right hand side is called entropy production density σ

$$\sigma = \sum_{r} \frac{\mathscr{A}_{r}}{T} w_{r} - \sum_{j} \boldsymbol{j}_{X_{j}} \cdot \boldsymbol{\nabla} \frac{\mu_{j}}{T}$$
(2.26)

In other words

$$\frac{ds_v}{dt} = \sigma \tag{2.27}$$

If we take the volume integral of equation (51) we get

$$\frac{dS}{dt} = \int \sigma dV \stackrel{(2.1)}{\ge} 0 \tag{2.28}$$

From here we get a local formulation of the second principle, i.e.

$$\sigma \ge 0 \tag{2.29}$$

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2.4. Beyond the Theorem of Minimum Entropy Production

A realistic description of chemical or analogous reaction requires an extension of the theory to the nonlinear range where the affinity is within the range of several kT's, i.e.

$$\mathscr{A} \ge kT \tag{2.30}$$

The entropy production density is (from equation (2.26))

$$\sigma = \sum_{r} \frac{\mathscr{A}_{r}}{T} w_{r} - \sum_{j} \boldsymbol{j}_{X_{j}} \cdot \boldsymbol{\nabla} \frac{\mu_{j}}{T}$$
(2.31)

We can now generalize the last equation by defining generalized forces, X and flows, J.

$$J_{j}^{diff} := \boldsymbol{j}_{X_{j}} \qquad X_{j}^{diff} := -\nabla \frac{\mu_{j}}{T}$$
$$J_{r}^{react} := w_{r} \qquad X_{r}^{react} := \mathscr{A}_{r} \qquad (2.32)$$

Then we get for the entropy production density the following expression

$$\sigma = \sum_{k} J_k X_k \tag{2.33}$$

Therefore the entropy production \dot{S} yields

$$\dot{S} = \int dV \sigma = \int dV \sum_{k} J_k X_k \tag{2.34}$$

Similar to the *Minimum Entropy Production* case we are interested in the stability. Therefore we investigate the time derivative of the entropy production \ddot{S}

$$\ddot{S} = \int dV \dot{\sigma} = \int dV \sum_{k} (\dot{J}_k X_k + J_k \dot{X}_k)$$
(2.35)

Beyond the linear range this expression does not exhibit any property of general validity. However in the linear range it leads to the *Minimum Entropy Production Theorem*

2.4.1. A general Criterion

We want now to extend the *Minimum Entropy Production Theorem* by coming back to the expression (2.32). By derivating with respect to time we get

$$\ddot{S} = \frac{1}{T} \int dV \frac{d}{dt} \left[\sum_{r} \mathscr{A}_{r} w_{r} - \sum_{j} \boldsymbol{j}_{X_{j}} \cdot \boldsymbol{\nabla} \mu_{j} \right] = \frac{1}{T} \int dV \left[\sum_{r} \dot{\mathscr{A}}_{r} w_{r} - \sum_{j} \frac{d}{dt} \left(\boldsymbol{j}_{X_{j}} \cdot \boldsymbol{\nabla} \mu_{j} \right) \right]$$
(2.36)

2.4. BEYOND THE THEOREM OF MINIMUM ENTROPY PRODUCTION

First we investigate the second term in the last equation, i.e.

$$\int dV \sum_{j} \frac{d}{dt} \left(\boldsymbol{j}_{X_{j}} \cdot \boldsymbol{\nabla} \mu_{j} \right) \stackrel{(2.22)}{=} \int dV \sum_{j} \frac{d}{dt} \left[\boldsymbol{\nabla} (\boldsymbol{j}_{X_{j}} \mu_{j}) - \mu_{j} \boldsymbol{\nabla} \cdot \boldsymbol{j}_{X_{j}} \right]$$
(2.37)

in the first term in the last equation we transform the integral into a surface integral, i.e.

$$\frac{d}{dt}\sum_{j}\underbrace{\int d\Sigma \boldsymbol{n}_{\Sigma} \cdot \boldsymbol{j}_{X_{j}}\mu_{j}}_{\text{for sufficient boundary conditions this yields 0}} - \int dV \sum_{j} \frac{d}{dt}(\mu_{j}\boldsymbol{\nabla} \cdot \boldsymbol{j}_{X_{j}}) = -\int dV \sum_{j} \frac{d}{dt}(\mu_{j}\boldsymbol{\nabla} \cdot \boldsymbol{j}_{X_{j}}) \quad (2.38)$$

with the product rule we get

$$-\int dV \sum_{j} \frac{d}{dt} (\mu_{j} \boldsymbol{\nabla} \cdot \boldsymbol{j}_{X_{j}}) = -\int dV \sum_{j} \left(\frac{\partial \mu_{j}}{\partial t} \boldsymbol{\nabla} \cdot \boldsymbol{j}_{X_{j}} + \mu_{j} \boldsymbol{\nabla} \cdot \frac{\partial \boldsymbol{j}_{X_{j}}}{\partial t} \right)$$
(2.39)

in the second term in the last equation we transform the integral into a surface integral, i.e.

$$-\int dV \sum_{j} \frac{d}{dt} (\mu_{j} \nabla \cdot \boldsymbol{j}_{X_{j}}) = -\int dV \sum_{j} \frac{\partial \mu_{j}}{\partial t} \nabla \cdot \boldsymbol{j}_{X_{j}} - \sum_{j} \underbrace{\int d\Sigma \boldsymbol{n}_{\Sigma} \cdot \frac{\partial \boldsymbol{j}_{X_{j}}}{\partial t} \mu_{j}}_{\text{for sufficient boundary conditions this yields 0}}$$
(2.40)

We are left with the following expression for the second term in equation (2.37|2)

$$\int dV \sum_{j} \frac{d}{dt} \left(\boldsymbol{j}_{X_{j}} \cdot \boldsymbol{\nabla} \boldsymbol{\mu}_{j} \right) = - \int dV \sum_{j} \frac{\partial \boldsymbol{\mu}_{j}}{\partial t} \boldsymbol{\nabla} \cdot \boldsymbol{j}_{X_{j}}$$
(2.41)

inserting this we get for \ddot{S}

$$\ddot{S} = \frac{1}{T} \int dV \left[\sum_{r} \dot{\mathscr{A}}_{r} w_{r} + \sum_{j} \frac{\partial \mu_{j}}{\partial t} \nabla \cdot \boldsymbol{j}_{X_{j}} \right] \stackrel{(2.20)}{=} \frac{1}{T} \int dV \left[-\sum_{r,k} w_{r} \nu_{r,k} \frac{\partial \mu_{k}}{\partial t} + \sum_{j} \frac{\partial \mu_{j}}{\partial t} \nabla \cdot \boldsymbol{j}_{X_{j}} \right]$$
(2.42)

now we can put $\sum_k rac{\partial \mu_k}{\partial t}$ as common factor in front and we get

$$\ddot{S} = \frac{1}{T} \int dV \sum_{k} \frac{\partial \mu_{k}}{\partial t} \underbrace{\left[-\sum_{r} w_{r} \nu_{r,k} + \nabla \cdot \boldsymbol{j}_{X_{k}} \right]}_{\text{from (2.18) this is } -\frac{\partial \rho_{X_{k}}}{\partial t}} = -\frac{1}{T} \int dV \sum_{k} \frac{\partial \mu_{k}}{\partial t} \frac{\partial \rho_{X_{k}}}{\partial t}$$
(2.43)

since μ_k is a function of $\{\rho_i\}_i$ we get

$$\ddot{S} = -\frac{1}{T} \int dV \sum_{k,r} \frac{\partial \mu_k}{\partial \rho_r} \frac{\partial \rho_r}{\partial t} \frac{\partial \rho_{X_k}}{\partial t}$$
(2.44)

It can be shown² that the quadratic form

$$\sum_{k,r} \frac{\partial \mu_k}{\partial \rho_r} \frac{\partial \rho_{X_r}}{\partial t} \frac{\partial \rho_{X_k}}{\partial t}$$
(2.45)

is positive definit. That result in

$$\ddot{S} \le 0 \tag{2.46}$$

This is the general criterion for the stability of far from equilibrium states.

2.5. Summary and Perspective

The aim of this paper was to give a general introduction into complex systems and hydrodynamic instabilites in the first chapter. In the second chapter the aim was to give an introduction to the theory of far from equilibrium thermodynamics and the connection to the formation of complex systems. Moreover I wanted to show that there exists a general criterion for which a nonequilibrium system is stable.

That a far from equilibrium system is stable is one important point but that a far from equilibrium system does form pattern is another. As a perspective I am going to investigate the need for the formation of pattern in a far from equilibrium state in connection to this general criterion.

²with the help of the inequalities of the thermodynamic potentials, similar to the linear range

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